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# ON THE REPRESENTATION AND THE RESIDUE OF CONCAVE FUNCTIONS

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**ABSTRACT.** In [2] we introduced several integral representation formulas for concave functions. Using those, we gave a general formula to describe the residue of concave functions with a pole at  $p \in (0, 1)$ . In the present article we will present alternate versions of the formulas, as well as a shortcut for the calculation to obtain the range of the residue.

**Key words:** concave univalent functions, integral representations

## 1. INTRODUCTION

Let  $\mathbb{C}$  be the complex plane,  $\hat{\mathbb{C}}$  the Riemann sphere and  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  be the unit disk. A univalent function  $f : \mathbb{D} \rightarrow \hat{\mathbb{C}}$  is said to be concave, if  $f(\mathbb{D})$  is concave, i.e.  $\mathbb{C} \setminus f(\mathbb{D})$  is convex. Commonly there are several types of concave functions, which map  $\mathbb{D}$  conformally onto a simply connected, concave domain in  $\hat{\mathbb{C}}$ :

- (1) meromorphic, univalent functions  $f$  with a simple pole at the origin and the normalization  $f(z) = \frac{1}{z} + \sum_{n=2}^{\infty} a_n z^n$ , said to belong to the class  $\mathcal{C}o_0$ ,
- (2) meromorphic, univalent functions  $f$  with a simple pole at the point  $p \in (0, 1)$  and the normalization  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ , said to belong to the class  $\mathcal{C}o_p$  and
- (3) analytic, univalent functions  $f$  satisfying  $f(1) = \infty$  with the normalizations  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$  and an opening angle of  $f(\mathbb{D})$  at  $\infty$  less or equal to  $\alpha\pi$  with  $\alpha \in (1, 2]$ , said to belong to the class  $\mathcal{C}o(\alpha)$ .

A detailed discussion of these classes has already been done in [2]. We therefore concentrate on the class  $\mathcal{C}o_p$  for the present article.

## 2. ALTERNATIVE FORMULAS

In [2] we introduced the following integral representation formula for functions of  $\mathcal{C}o_p$ .

**Theorem 1.** [2] *Let  $p \in (0, 1)$ . For a meromorphic function  $f : \mathbb{D} \rightarrow \hat{\mathbb{C}}$  of class  $\mathcal{C}o_p$ , there exists a function  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ , holomorphic in  $\mathbb{D}$  with  $\varphi(p) = p$ , such that the concave function can be represented as*

$$(1) \quad f'(z) = \frac{p^2}{(z-p)^2(1-zp)^2} \exp \int_0^z \frac{-2\varphi(\zeta)}{1-\zeta\varphi(\zeta)} d\zeta$$

*for  $z \in \mathbb{D}$ . Conversely, for any holomorphic function  $\varphi$  mapping  $\mathbb{D} \rightarrow \mathbb{D}$  with  $\varphi(p) = p$ , there exists a concave function of class  $\mathcal{C}o_p$  described by (1).*

However, a fixed point of the function  $\varphi$  at  $p$  is not very useful for further discussions. Using several transformations we obtain an alternate version of Theorem 1.

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**Corollary 2.** Let  $p \in (0, 1)$ . For a meromorphic function  $f : \mathbb{D} \rightarrow \hat{\mathbb{C}}$  of class  $Co_p$ , there exists a function  $\Psi : \mathbb{D} \rightarrow \mathbb{D}$ , holomorphic in  $\mathbb{D}$  with  $\Psi(0) = 0$  such that the concave function can be represented as

$$(2) \quad f'(z) = \frac{p^2}{(z-p)^2(1-zp)^2} \exp \left( 2 \int_p^{\frac{p-z}{1-pz}} \frac{p}{1-p\zeta} - \frac{\Psi(\zeta)}{1-\zeta\Psi(\zeta)} d\zeta \right)$$

for  $z \in \mathbb{D}$ . Conversely, for any holomorphic function  $\Psi$  mapping  $\mathbb{D} \rightarrow \mathbb{D}$  with  $\Psi(0) = 0$ , there exists a concave function of class  $Co_p$  described by (2).

*Proof.* Let  $p \in (0, 1)$  and  $z \in \mathbb{D}$ . Applying the transformation  $\zeta = \frac{p-x}{1-px}$  and  $\Phi(x) = \varphi(\zeta)$  we obtain

$$\begin{aligned} \int_0^z \frac{-2\varphi(\zeta)}{1-\zeta\varphi(\zeta)} d\zeta &= \int_p^{\frac{p-z}{1-pz}} \frac{-2\Phi(x)}{1-\frac{p-x}{1-px}\Phi(x)} \cdot \frac{p^2-1}{(1-px)^2} dx \\ &= \int_p^{\frac{p-z}{1-pz}} \frac{-2\Phi(x)(p^2-1)}{(1-px)^2 - (p-x)\Phi(x)(1-px)} dx. \end{aligned}$$

Here the function  $\Phi$  is holomorphic in  $\mathbb{D}$  with  $\Phi(0) = p$ . Therefore there exists a function  $\Psi : \mathbb{D} \rightarrow \mathbb{D}$  holomorphic in  $\mathbb{D}$  with  $\Psi(0) = 0$ , such that  $\Phi(x) = \frac{p-\Psi(x)}{1-p\Psi(x)}$ . Then

$$\begin{aligned} \int_0^z \frac{-2\varphi(\zeta)}{1-\zeta\varphi(\zeta)} d\zeta &= \int_p^{\frac{p-z}{1-pz}} \frac{-2\frac{p-\Psi(x)}{1-p\Psi(x)}(p^2-1)}{(1-px)^2 - (p-x)\frac{p-\Psi(x)}{1-p\Psi(x)}(1-px)} dx \\ &= \int_p^{\frac{p-z}{1-pz}} \frac{-2(p-\Psi(x))(p^2-1)}{(1-px)((1-p^2)-x\Psi(x)(1-p^2))} dx \\ &= \int_p^{\frac{p-z}{1-pz}} \frac{-2(\Psi(x)-p)}{(1-px)(1-x\Psi(x))} dx \\ &= 2 \int_p^{\frac{p-z}{1-pz}} \frac{p}{1-px} - \frac{\Psi(x)}{1-x\Psi(x)} dx. \end{aligned}$$

Changing the variable inside the integration and replacing the integral in (1) leads to the statement.  $\square$

The formula for the residue derived from the integral representation in [2] was given as follows.

**Theorem 3.** [2] Let  $f(z) \in Co_p$  be a concave function with a simple pole at some point  $p \in (0, 1)$ . Then the residue of this function  $f$  can be described by some function  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ , holomorphic in  $\mathbb{D}$  and  $\varphi(p) = p$ , such that

$$(3) \quad \text{Res}_p f = -\frac{p^2}{(1-p^2)^2} \exp \int_0^p \frac{-2\varphi(z)}{1-x\varphi(z)} dz.$$

Applying the alternative representation from Corollary 2, we obtain

**Corollary 4.** Let  $f(z) \in Co_p$  be a concave function with a simple pole at some point  $p \in (0, 1)$ . Then the residue of this function  $f$  can be described by some function  $\Psi : \mathbb{D} \rightarrow \mathbb{D}$ , holomorphic in  $\mathbb{D}$  and  $\Psi(0) = 0$ , such that

$$(4) \quad \text{Res}_p f = -\frac{p^2}{(1-p^2)^2} \exp 2 \int_0^p \frac{\Psi(x)}{1-x\Psi(x)} - \frac{p}{1-px} dx.$$

The advantage of Corollary 4 over the original presentation is the fixed point of  $\Psi$  at the origin. This provide much easier means for the construction, than a fixed point at  $p$ . Furthermore, the Schwarz Lemma can be applied directly without any complicated analysis, giving a way for the estimate of special values. We will show an application in the next section.

### 3. RANGE OF THE RESIDUE

Wirhth proved the following statement in [3] using the inequality

$$\left| \frac{1}{f(z)} - \frac{1}{z} + \frac{1+p^2}{p} \right| \leq 1$$

provided by Miller in [1].

**Theorem 5.** [3] *Let  $p \in (0, 1)$ . For  $a \in \mathbb{C}$  there exists a function  $f \in Co_p$  such that  $a = Res_p f$  if and only if*

$$(5) \quad \left| a + \frac{p^2}{1-p^4} \right| \leq \frac{p^4}{1-p^4}.$$

Let  $\vartheta \in [0, 2\pi)$ . A function  $f \in Co_p$  has the residue

$$a = -\frac{p^2}{1-p^4} + e^{i\vartheta} \frac{p^4}{1-p^4}$$

if and only if

$$(6) \quad f_{\vartheta}(z) = \frac{z - \frac{p}{1+p^2}(1 + e^{i\vartheta})z^2}{\left(1 - \frac{z}{p}\right)(1 - pz)}.$$

The established representation formula for the residue can be used for a different approach of the same statement as described in [2]. For the present discussion we will use Corollary 4, which provides a shortcut for the proof. We also present some details, omitted in [2]

*Proof.* Let  $p \in (0, 1)$  and  $\Psi : \mathbb{D} \rightarrow \mathbb{D}$  be holomorphic in  $\mathbb{D}$  with fixed point at the origin. For  $a = Res_p f$  with  $f \in Co_p$  we obtain with the use of Corollary 4

$$\left| a + \frac{p^2}{1-p^4} \right| \stackrel{(4)}{=} \frac{p^2}{1-p^4} \left| \frac{1+p^2}{1-p^2} \exp \left( 2 \int_0^p \frac{\Psi(x)}{1-x\Psi(x)} - \frac{p}{1-px} dx \right) - 1 \right|.$$

Some basic calculations yield

$$\frac{1+p^2}{1-p^2} = \exp 2 \left( \frac{1}{2} \log \frac{1+p^2}{1-p^2} \right) = \exp \int_0^p \frac{2p}{1-p^2x^2} dx$$

and therefore

$$\begin{aligned} \left| a + \frac{p^2}{1-p^4} \right| &= \frac{p^2}{1-p^4} \left| \exp \int_0^p 2 \left( \frac{p}{1-p^2x^2} - \frac{p}{1-px} + \frac{\Psi(x)}{1-x\Psi(x)} \right) dx - 1 \right| \\ &= \frac{p^2}{1-p^4} \left| \exp \int_0^p 2 \frac{\Psi(x) - p^2x}{(1-x\Psi(x))(1-p^2x^2)} dx - 1 \right|. \end{aligned}$$

From the triangle inequality, we know that

$$|e^w - 1| = \left| \sum_{n=1}^{\infty} \frac{w^n}{n!} \right| \leq \sum_{n=1}^{\infty} \frac{|w|^n}{n!} = e^{|w|} - 1.$$

Hence

$$\left| a + \frac{p^2}{1-p^4} \right| \leq \frac{p^2}{1-p^4} \left( \exp \int_0^p 2 \left| \frac{\Psi(x) - p^2x}{(1-x\Psi(x))(1-p^2x^2)} \right| dx - 1 \right).$$

Due to the fixed point at the origin, we can apply the Schwarz Lemma and have  $|\Psi(x)| \leq x$  for  $0 < x < p$ . Furthermore, since  $\left| \frac{w-p^2x}{1-xw} \right| \leq \frac{(1-p^2)x}{1-x^2}$  for  $|w| \leq x$ , we have

$$(7) \quad \left| \frac{\Psi(x) - p^2x}{1-x\Psi(x)} \right| \leq \frac{(1-p^2)x}{1-x^2}.$$

Using the above, we finally obtain

$$\begin{aligned} \left| a + \frac{p^2}{1-p^4} \right| &\stackrel{(7)}{\leq} \frac{p^2}{1-p^4} \left( \exp \int_0^p 2 \frac{(1-p^2)x}{(1-x^2)(1-p^2x^2)} dx - 1 \right) \\ &= \frac{p^2}{1-p^4} (\exp(\log(1+p^2)) - 1) \\ &= \frac{p^4}{1-p^4}. \end{aligned}$$

The rest of the proof goes according to the way described in [2].

□

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